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Some Properties of the Dirichlet Type Integrals

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Abstract – **This article considers the Dirichlet type integrals, for which closed form formulas have been obtained. In particular, it is proven that all these integrals are rational numbers. Feynman's method for integral calculating is also used in the work. The integrals considered are related to a remarkable Lobachevsky formula, described at the end.**

Keywords – Dirichlet integral, Feynman integration method, Lobachevsky integral formula.

I. Introduction

Dirichlet integral [1], [2]

$$\mathcal{D} = \int_0^\infty \frac{\sin x}{x} dx$$

has a key role in the theory of the Fourier transform [3], [4], [5].

This integral is an important (perhaps the most important) example of a conditionally convergent integral [6]. We have

$$\int_0^\infty \left|\frac{\sin x}{x}\right| dx = \infty$$

because

$$\int_0^{n\pi} \left|\frac{\sin x}{x}\right| dx \geq \frac{2}{\pi}\sum_{k=1}^{n}\frac{1}{k} \approx \frac{2}{\pi}\ln n \to \infty$$

Any interesting property of this integral is supposed to be the subject of special attention with a presumed eventual application in signal processing theory.

II. Matherials and Methods

This paper offers mainly theoretical content. The results are additionally verified empirically by examining many particular cases using specialized software for analytic transformations. Feynman's famous (brilliant) method for calculating some parametric integrals is also used here [7], [8], which offers an original way to estimate $\mathcal{D}$. Also, in the text below various calculus technics are used, including the classic Laplace Transform.

The value of $\mathcal{D}$ is $\pi/2$, which can be shown in various ways, one of which is related to the Feynman approach. Put

$$J(a) = \int_0^\infty \frac{\sin x}{x} e^{-ax} dx$$

where $a \geq 0$ is a parameter. Then we have

$$J'(a) = \int_0^\infty \frac{\sin x}{x} e^{-ax}(-x) dx$$

$$= -\int_0^\infty \sin x \, e^{-ax} dx = -\frac{1}{1+a^2}$$

from which we find

$$J(a) = -\int \frac{da}{1+a^2} + const = -\arctan a + const$$

On the other hand, $J(\infty) = 0$, therefore

$$0 = -\arctan \infty + const$$

$$0 = -\frac{\pi}{2} + const \Rightarrow const = \frac{\pi}{2}$$

which gives

$$J(a) = -\arctan a + \frac{\pi}{2}$$

$$\mathcal{D} = J(0) = \arctan 0 + \frac{\pi}{2} = \frac{\pi}{2}$$

III. Results and Discussion

The main result of this article is contained in the following theorem.

Theorem 1. Consider the Dirichlet type integrals

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$$\mathcal{D}_n = \frac{1}{\pi}\int_0^\infty \left(\frac{\sin x}{x}\right)^n dx\,, n \in \mathbb{N}$$

1) Let $n = 2m + 1$, $m \in \mathbb{N}$, be an odd number. Put $z_k = -(2k+1)^2, 0 \leq k \leq m,$

$$f(z) = (z - z_0)(z - z_1)\cdot \cdots \cdot (z - z_m)$$

Then

$$\mathcal{D}_n = \frac{n}{2}\sum_{k=0}^{m}\left(\frac{z_k^m}{f'(z_k)}\right)\frac{1}{(2k+1)}$$

2) Let $n = 2m$, $m \in \mathbb{N}$, be an even number. Put $z_k = -(2k)^2, 1 \leq k \leq m,$

$$f(z) = (z - z_1)(z - z_2)\cdot \cdots \cdot (z - z_m)$$

Then

$$\mathcal{D}_n = \frac{n}{4}\sum_{k=1}^{m}\left(\frac{z_k^{m-1}}{f'(z_k)}\right)\frac{1}{k}$$

3) For every natural number $n \in \mathbb{N}$, the value of the integral $\mathcal{D}_n$ is a rational number.

We will give the proof of Theorem 1 right below. In particular

$$\mathcal{D}_1 = \frac{1}{\pi}\int_0^\infty \frac{\sin x}{x} dx = \frac{1}{2}, \mathcal{D}_2 = \frac{1}{\pi}\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{1}{2}$$

$$\mathcal{D}_3 = \frac{1}{\pi}\int_0^\infty \left(\frac{\sin x}{x}\right)^3 dx = \frac{3}{8}$$

$$\mathcal{D}_4 = \frac{1}{\pi}\int_0^\infty \left(\frac{\sin x}{x}\right)^4 dx = \frac{1}{3}$$

$$\mathcal{D}_{10} = \frac{1}{\pi}\int_0^\infty \left(\frac{\sin x}{x}\right)^{10} dx = \frac{15619}{72576}$$

$$\mathcal{D}_{20} = \frac{1}{\pi}\int_0^\infty \left(\frac{\sin x}{x}\right)^{20} dx = \frac{37307713155613}{243290200817664}$$

$$\mathcal{D}_{30} = \frac{1}{\pi}\int_0^\infty \left(\frac{\sin x}{x}\right)^{30} dx = \frac{18497593486903125823791655511}{147362699895661699242393600000}$$

All $\mathcal{D}_n$ are absolutely convergent for $n \geq 2$.

PROOF OF THEOREM 1. Hereafter $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ are the traditional notations for the natural, the rational and the real numbers respectively. Start with a proposition, the validity of which follows from the properties of the Laplace transform.

Proposition 1. For every natural number $n \in \mathbb{N}$ it holds

$$\int_0^\infty s^{n-1}e^{-sx}ds = \frac{\Gamma(n)}{x^n} = \frac{(n-1)!}{x^n}, x > 0$$

where $\Gamma(\cdot)$ is the Euler gamma function. ■

Proposition 1 allows to represent the integrals $\mathcal{D}_n$ in the form

$$\mathcal{D}_n = \frac{1}{\pi(n-1)!}\int_0^\infty \sin^n x\left(\int_0^\infty s^{n-1}e^{-sx}ds\right)dx$$

After changing the order of integration, we obtain

$$\mathcal{D}_n = \frac{1}{\pi(n-1)!}\int_0^\infty s^{n-1}\left(\int_0^\infty \sin^n x\, e^{-sx}dx\right)ds \quad (1)$$

The validity of the following proposition is a consequence from the representation of a proper rational function as a sum of elementary fractions.

Proposition 2. Let $Q(z)$be a polynomial with $\deg Q \leq m-1$, $m \in \mathbb{N}$, and $z_k \in \mathbb{C}$, $1 \leq k \leq m$, are distinct numbers. Let also

$$f(z) = (z - z_1)\cdot \cdots \cdot (z - z_m)$$

Then

$$\frac{Q(z)}{f(z)} = \sum_{k=1}^{m}\frac{a_k}{z - z_k}, a_k = \frac{Q(z_k)}{f'(z_k)}$$

Therefore, if the polynomial $Q(z)$ has rational coefficients and $z_k \in \mathbb{Q}$ are rational numbers, then the numbers a_k are also rational. ■

The following statement plays a very basic role in the proof of Theorem 1.

Proposition 3. Let

$$S_n = s^{n-1}\int_0^\infty (\sin^n x)e^{-sx}dx\,, n \in \mathbb{N}, s > 0$$

Then it holds

$$S_{n+2} = S_n\frac{(n+1)(n+2)}{((n+2)^2 + s^2)}$$

Therefore, if $= 2m + 1$, $m \in \mathbb{N}$, is an odd number, then

$$S_n = \frac{n!\,(s^2)^m}{(s^2+1)(s^2+9)\cdot \cdots \cdot (s^2+n^2)}$$

and if $n = 2m$, $m \in \mathbb{N}$, is an even number, then

$$S_n = \frac{n!\,(s^2)^{m-1}}{(s^2+4)(s^2+16)\cdot \cdots \cdot (s^2+n^2)}$$

In the first case, the degree of the denominator as a function of s^2 is $m + 1$, and in the second case is m. In both cases, S_n is a proper rational function of s^2.

Proof. The proof is based on successive transformations, using the technique of integration by parts.

$$S_{n+2} = \int_0^\infty (\sin^{n+2} x) e^{-sx} dx$$
$$= \int_0^\infty (\sin^n x)(1-\cos^2 x) e^{-sx} dx$$
$$= S_n - \int_0^\infty (\sin^n x)\cos^2 x\, e^{-sx} dx$$
$$= S_n - \int_0^\infty (\sin^n x)\cos x\, e^{-sx} d\sin x$$
$$= S_n - \frac{1}{n+1}\int_0^\infty \cos x\, e^{-sx} d\sin^{n+1} x$$
$$= S_n + \frac{1}{n+1}\int_0^\infty \sin^{n+1} x\, d\cos x\, e^{-sx}$$
$$= S_n + \frac{1}{n+1}\int_0^\infty (\sin^{n+1} x)(-\sin x\, e^{-sx} - s\cos x\, e^{-sx}) dx$$
$$= S_n + \frac{1}{n+1}\int_0^\infty (-\sin^{n+2} x\, e^{-sx} - s\sin^{n+1} x\cos x\, e^{-sx}) dx$$
$$= S_n - \frac{1}{n+1}\int_0^\infty \sin^{n+2} x\, e^{-sx} dx - \frac{s}{n+1}\int_0^\infty (\sin^{n+1} x\cos x\, e^{-sx}) dx$$
$$= S_n - \frac{S_{n+2}}{n+1} - \frac{s}{n+1}\int_0^\infty \sin^{n+1} x\, e^{-sx} d\sin x$$
$$= S_n - \frac{S_{n+2}}{n+1} - \frac{s}{(n+1)(n+2)}\int_0^\infty e^{-sx} d\sin^{n+2} x$$
$$= S_n - \frac{S_{n+2}}{n+1} + \frac{s}{(n+1)(n+2)}\int_0^\infty \sin^{n+2} x\, de^{-sx}$$
$$= S_n - \frac{S_{n+2}}{n+1} + \frac{s}{(n+1)(n+2)}\int_0^\infty (\sin^{n+2} x)(-se^{-sx}) dx$$
$$= S_n - \frac{S_{n+2}}{n+1} - \frac{s^2}{(n+1)(n+2)}\int_0^\infty (\sin^{n+2} x)(e^{-sx}) dx$$
$$= S_n - \frac{S_{n+2}}{n+1} - \frac{s^2}{(n+1)(n+2)} S_{n+2}$$

Therefore

$$S_{n+2} = S_n - \frac{S_{n+2}}{n+1} - \frac{s^2}{(n+1)(n+2)} S_{n+2}$$

which implies the proof of Proposition 3. ■

Now we are at position to finish the proof of Theorem 1. Suppose that $n = 2m+1$, $m \in \mathbb{N}$, is an odd number and put $s^2 = z$, $z_k = -(2k+1)^2$, $0 \le k \le m$, $Q(z) = z^m$

$$f(z) = (z-z_0)(z-z_1)\cdot\cdots\cdot(z-z_m)$$

From Proposition 2 we obtain

$$S_n = \frac{n!\, z^m}{f(z)} = n!\sum_{k=0}^{m}\left(\frac{Q(z_k)}{f'(z_k)}\right)\frac{1}{z-z_k} = n!\sum_{k=0}^{m}\left(\frac{z_k^m}{f'(z_k)}\right)\frac{1}{s^2+(2k+1)^2}$$

According to (1)

$$\mathcal{D}_n = \frac{1}{\pi(n-1)!}\int_0^\infty S_n ds = \frac{n!}{\pi(n-1)!}\int_0^\infty \sum_{k=0}^{m}\left(\frac{z_k^m}{f'(z_k)}\right)\frac{1}{s^2+(2k+1)^2} ds$$

$$\mathcal{D}_n = \frac{n}{\pi}\sum_{k=0}^{m}\left(\frac{z_k^m}{f'(z_k)}\right)\int_0^\infty \frac{1}{s^2+(2k+1)^2} ds$$

$$\mathcal{D}_n = \frac{n}{\pi}\sum_{k=0}^{m}\left(\frac{z_k^m}{f'(z_k)}\right)\left(\frac{\pi}{2(2k+1)}\right)$$

$$\mathcal{D}_n = \frac{n}{2}\sum_{k=0}^{m}\left(\frac{z_k^m}{f'(z_k)}\right)\frac{1}{(2k+1)} \Longrightarrow \mathcal{D}_n \in \mathbb{Q} \quad (2)$$

Now let $n = 2m$, $m \in \mathbb{N}$, be an even number and let $s^2 = z$, $z_k = -(2k)^2$, $1 \le k \le m$, $Q(z) = z^{m-1}$

$$f(z) = (z-z_1)(z-z_2)\cdot\cdots\cdot(z-z_m)$$

Then from Proposition 2 we obtain

$$S_n = \frac{n!\, z^{m-1}}{f(z)} = n!\sum_{k=1}^{m}\left(\frac{Q(z_k)}{f'(z_k)}\right)\frac{1}{z-z_k} = n!\sum_{k=1}^{m}\left(\frac{z_k^{m-1}}{f'(z_k)}\right)\frac{1}{s^2+(2k)^2}$$

Again, according to (1)

$$\mathcal{D}_n = \frac{1}{\pi(n-1)!}\int_0^\infty S_n ds = \frac{n!}{\pi(n-1)!}\int_0^\infty \sum_{k=1}^{m}\left(\frac{z_k^{m-1}}{f'(z_k)}\right)\frac{1}{s^2+(2k)^2} ds$$

$$\mathcal{D}_n = \frac{n}{\pi}\sum_{k=1}^{m}\left(\frac{z_k^{m-1}}{f'(z_k)}\right)\int_0^\infty \frac{1}{s^2+(2k)^2} ds = \frac{n}{\pi}\sum_{k=1}^{m}\left(\frac{z_k^{m-1}}{f'(z_k)}\right)\left(\frac{\pi}{4k}\right)$$

$$\mathcal{D}_n = \frac{n}{4}\sum_{k=1}^{m}\left(\frac{z_k^{m-1}}{f'(z_k)}\right)\frac{1}{k} \Longrightarrow \mathcal{D}_n \in \mathbb{Q} \qquad (3)$$

Relations (2) and (3) prove completely Theorem 1. ■

The authors of the present work perceive as an amazing fact the rational value of $\mathcal{D}_n \in \mathbb{Q}$ for any $n \in \mathbb{N}$. The uniqueness and significance of the fundamental integral $\mathcal{D}$ contains sufficient motivation to search various regularities related to it, revealing at the same time various discussion opportunities.

IV. Lobachevski Integral Formula

The integrals considered above are related to a remarkable formula, the content of which is given in the following theorem, presented for example in [9] and [10].

Theorem 2. Let the continuous (or locally integrable) function $f(x)\colon \mathbb{R} \to \mathbb{R}$ satisfy the condition

$$f(\pi + x) = f(\pi - x) = f(x)\ \forall x \in \mathbb{R}$$

Then

$$\int_0^\infty f(x)\frac{\sin^2 x}{x^2}dx = \int_0^\infty f(x)\frac{\sin x}{x}dx = \int_0^{\frac{\pi}{2}} f(x)dx \ ■$$

By condition, the function $f(x)$ is even, because

$$f(-x) = f\big(\pi - (-x)\big) = f(\pi + x) = f(x)$$

and also π-periodic. A detailed proof of Theorem 2 can be found, for instance, in [9]. but the proof assumes a solid knowledge from an undergraduate course in mathematics. In particular, from Theorem 2 it follows that

$$\int_0^\infty \frac{\sin^2 x}{x^2}dx = \int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$$

V. Conclusions

The improper conditionally convergent Dirichlet integral is always of great interest, as it is related to the convergence of the Fourier transform. This work describes approaches to some analytic calculations, for which we modestly hope may appear to be useful also in other circumstances, potentially concerning digital signal processing.

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